$$\begin{aligned}\nabla\cdot\mathbf{r}f(r) &= 3f(r)+\frac{x^2}{r}\frac{\mathrm{d}f}{\mathrm{d}r}+\frac{y^2}{r}\frac{\mathrm{d}f}{\mathrm{d}r}+\frac{z^2}{r}\frac{\mathrm{d}f}{\mathrm{d}r}\\ &= 3f(r)+\frac{x^2+y^2+z^2}{r}\frac{\mathrm{d}f}{\mathrm{d}r}=3f(r)+r\frac{\mathrm{d}f}{\mathrm{d}r}.\end{aligned}$$

## 1.11 Divergence Theorem

The *divergence theorem* is also known as *Gauss' theorem*. It relates the flux of a vector field through a closed surface $S$ to the divergence of the vector field in the enclosed volume

$$\boxed{\iint_{\text{closed surface } S}\mathbf{A}\cdot\mathbf{n}\,\mathrm{d}a=\iiint_{\text{vol. enclosed in } S}\nabla\cdot\mathbf{A}\,\mathrm{d}V.} \qquad (2.72)$$

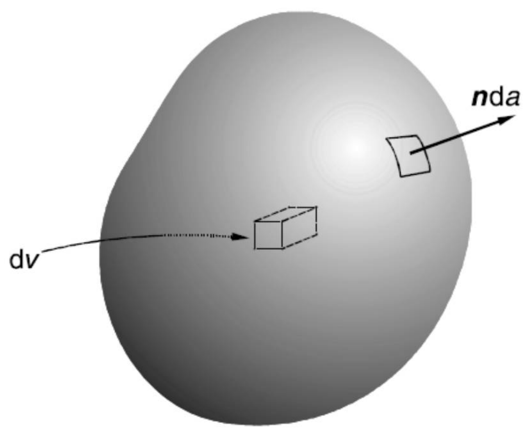


**Fig. 2.11.** The divergence theorem. The volume is enclosed by the surface. The integral of the divergence over the volume inside is equal to the flux through the outside surface

## 1.12 Continuity Equation

One of the most important applications of the divergence theorem is using it to express the conservation laws in differential forms. As an example, consider a fluid of density $\varrho$ moving with velocity $\mathbf{v}$. According to (2.70), the rate at which the fluid flows out of a closed surface is

$$\text{Rate of outward flow through a closed surface} = \oiint_S \varrho\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a. \tag{2.76}$$

Now because of the conservation of mass, this rate of out flow must be equal to the rate of decrease of the fluid inside the volume that is enclosed by the surface. Therefore

$$\oiint_S \varrho\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a = -\iiint_V \frac{\partial\varrho}{\partial t}\mathrm{d}V. \tag{2.77}$$

The negative sign accounts for the fact that the fluid inside is decreasing if the flow is outward. Using the divergence theorem

$$\oiint_S \varrho\mathbf{v}\cdot\mathbf{n}\,\mathrm{d}a = \iiint_V \boldsymbol{\nabla}\cdot(\varrho\mathbf{v})\,\mathrm{d}V, \tag{2.78}$$

we have

$$\iiint_V \left[\boldsymbol{\nabla}\cdot(\varrho\mathbf{v}) + \frac{\partial\varrho}{\partial t}\right]\mathrm{d}V = 0. \tag{2.79}$$

Since the volume $V$ in this equation, the integrand must equal to zero, or

$$\boxed{\boldsymbol{\nabla}\cdot(\varrho\mathbf{v}) + \frac{\partial\varrho}{\partial t} = 0.} \tag{2.80}$$

## 1.13 The Curl of a Vector

The cross product of the gradient operator $\boldsymbol{\nabla}$ with vector $\mathbf{A}$ gives us another special combination of the derivatives of the components of $\mathbf{A}$

$$\boldsymbol{\nabla}\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right). \tag{2.82}$$


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It is a vector known as the *curl* of $\mathbf{A}$. The name curl suggests that it has something to do with rotation. In fact, in European texts the word rotation (or rot) is used in place of curl. In Example 2.1.3, we have considered the motion of a body rotating around the $z$-axis with angular velocity $\boldsymbol{\omega}$. The velocity of the particles in the body is $\mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$. The circular characteristic of this velocity field is manifested in the curl of the velocity

$$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\omega\mathbf{k}, \tag{2.83}$$

If the curl of a vector field is equal to zero everywhere, the field is called *irrotational.*

*Example 2.6.1.* Show that $(a)$ $\nabla \times \mathbf{r} = \mathbf{0}$; $(b)$ $\nabla \times \mathbf{r}f(r) = \mathbf{0}$ where $\mathbf{r}$ is the position vector.

**Solution 2.6.1.** $(a)$ Since $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, so

$$\nabla \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = \mathbf{0}.$$

$(b)$

$$\begin{aligned} \nabla \times \mathbf{r}f(r) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xf(r) & yf(r) & zf(r) \end{vmatrix} = \left\{\frac{\partial}{\partial y}[zf(r)] - \frac{\partial}{\partial z}[yf(r)]\right\}\mathbf{i} \\ &\quad + \left\{\frac{\partial}{\partial z}[xf(r)] - \frac{\partial}{\partial x}[zf(r)]\right\}\mathbf{j} + \left\{\frac{\partial}{\partial x}[yf(r)] - \frac{\partial}{\partial y}[xf(r)]\right\}\mathbf{k} \\ &= \left\{z\frac{\partial}{\partial y}f(r) - y\frac{\partial}{\partial z}f(r)\right\}\mathbf{i} + \left\{x\frac{\partial}{\partial z}f(r) - z\frac{\partial}{\partial x}f(r)\right\}\mathbf{j} \\ &\quad + \left\{y\frac{\partial}{\partial x}f(r) - x\frac{\partial}{\partial y}f(r)\right\}\mathbf{k}. \end{aligned}$$

Since

$$\frac{\partial}{\partial y} f(r) = \frac{\mathrm{d}f}{\mathrm{d}r}\frac{\partial r}{\partial y} \text{ and } r = \left(x^2+y^2+z^2\right)^{1/2},$$

$$\frac{\partial}{\partial y} f(r) = \frac{\mathrm{d}f}{\mathrm{d}r}\left(-\frac{y}{\left(x^2+y^2+z^2\right)^{1/2}}\right) = -\frac{\mathrm{d}f}{\mathrm{d}r}\frac{y}{r},$$

$$\frac{\partial}{\partial x} f(r) = -\frac{\mathrm{d}f}{\mathrm{d}r}\frac{x}{r}; \qquad \frac{\partial}{\partial z} f(r) = -\frac{\mathrm{d}f}{\mathrm{d}r}\frac{z}{r}.$$

Therefore

$$\nabla \times \mathbf{r} f(r) = \frac{\mathrm{d}f}{\mathrm{d}r}\left\{-z\frac{y}{r}+y\frac{z}{r}\right\}\mathbf{i}+\frac{\mathrm{d}f}{\mathrm{d}r}\left\{-x\frac{z}{r}+z\frac{x}{r}\right\}\mathbf{j}+\frac{\mathrm{d}f}{\mathrm{d}r}\left\{-y\frac{x}{r}+x\frac{y}{r}\right\}\mathbf{k}=\mathbf{0}.$$

## 1.14 Stokes' Theorem

*Stokes' theorem* relates the line integral of a vector function around a closed loop $C$ to a surface integral of the curl of that vector over a surface $S$ that spans the loop. The theorem states that

$$\boxed{\int_{\text{closed loop } C} \mathbf{A}\cdot \mathrm{d}\mathbf{r} = \iint_{\text{area bounded by } C} (\nabla \times \mathbf{A})\cdot \mathbf{n}\, \mathrm{d}a,} \qquad (2.84)$$

where $\mathbf{dr}$ is a directed line element along a closed curve $C$ and $S$ is any surface bounded by $C$. At any point on the surface, the unit normal vector $\mathbf{n}$ is perpendicular to the surface element $\mathrm{d}a$ at that point as shown in Fig. 2.14.

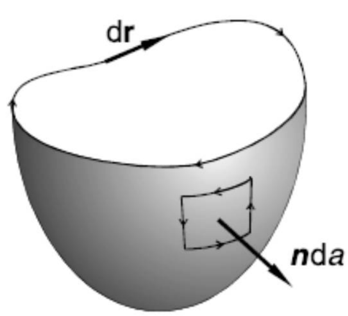


**Fig. 2.14.** Stokes' theorem. The integral of the curl over the surface is equal to the line integral around the closed boundary curve

*Example 2.6.4.* Use Stokes' theorem to evaluate the line integral $\oint_C \mathbf{A}\cdot d\mathbf{r}$ with $\mathbf{A} = 2yz\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ along the circle described by $x^2 + y^2 = 1$.

**Solution 2.6.4.** The curl of $\mathbf{A}$ is

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2yz & x & z^2 \end{vmatrix} = 2y\mathbf{j} + (1-2z)\mathbf{k},$$

and according to Stokes' theorem

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{A}\cdot \mathbf{n}\, da = \iint_S [2y\mathbf{j} + (1-2z)\mathbf{k}]\cdot \mathbf{n}\, da.$$

Since $S$ can be any surface as long as it is bounded by the circle, the simplest way to do this problem is to use the flat surface inside the circle. In that case $z = 0$ and $\mathbf{n} = \mathbf{k}$. Hence,

$$\oint \mathbf{A}\cdot d\mathbf{r} = \iint_S da = \pi.$$

$$\nabla \times \nabla\varphi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\varphi}{\partial x} & \frac{\partial\varphi}{\partial y} & \frac{\partial\varphi}{\partial z} \end{vmatrix} = \mathbf{i}\left(\frac{\partial^2\varphi}{\partial y\partial z} - \frac{\partial^2\varphi}{\partial z\partial y}\right)$$
$$+\mathbf{j}\left(\frac{\partial^2\varphi}{\partial z\partial x} - \frac{\partial^2\varphi}{\partial x\partial z}\right) + \mathbf{k}\left(\frac{\partial^2\varphi}{\partial x\partial y} - \frac{\partial^2\varphi}{\partial y\partial x}\right) = \mathbf{0}, \qquad (2.109)$$


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$$\nabla\cdot\nabla\times\mathbf{A}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ A_x & A_y & A_z\end{vmatrix}=0. \tag{2.110}$$



*Example 2.7.3.* If

$$\nabla\cdot\mathbf{E}=0,\qquad \nabla\times\mathbf{E}=-\frac{\partial}{\partial t}\mathbf{H},$$

$$\nabla\cdot\mathbf{H}=0,\qquad \nabla\times\mathbf{H}=\frac{\partial}{\partial t}\mathbf{E},$$

show that

$$\nabla^2\mathbf{E}=\frac{\partial^2}{\partial t^2}\mathbf{E};\quad \nabla^2\mathbf{H}=\frac{\partial^2}{\partial t^2}\mathbf{H}.$$

**Solution 2.7.3.**

$$\nabla\times(\nabla\times\mathbf{E})=\nabla\times\left(-\frac{\partial}{\partial t}\mathbf{H}\right)=-\frac{\partial}{\partial t}(\nabla\times\mathbf{H})=-\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\mathbf{E}\right)=-\frac{\partial^2}{\partial t^2}\mathbf{E},$$

$$\nabla\times(\nabla\times\mathbf{E})=\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E}=-\nabla^2\mathbf{E}\quad(\text{since }\nabla\cdot\mathbf{E}=0).$$

Therefore

$$\nabla^2\mathbf{E}=\frac{\partial^2}{\partial t^2}\mathbf{E}.$$

Similarly,

$$\nabla\times(\nabla\times\mathbf{H})=\nabla\times\left(\frac{\partial}{\partial t}\mathbf{E}\right)=\frac{\partial}{\partial t}(\nabla\times\mathbf{E})=\frac{\partial}{\partial t}\left(-\frac{\partial}{\partial t}\mathbf{H}\right)=-\frac{\partial^2}{\partial t^2}\mathbf{H},$$

$$\nabla\times(\nabla\times\mathbf{H})=\nabla(\nabla\cdot\mathbf{H})-\nabla^2\mathbf{H}=-\nabla^2\mathbf{H}\quad(\text{since }\nabla\cdot\mathbf{H}=0).$$

It follows that

$$\nabla^2\mathbf{H}=\frac{\partial^2}{\partial t^2}\mathbf{H}.$$